Lecture 2-1 Scaling Law

Sizes of objects:

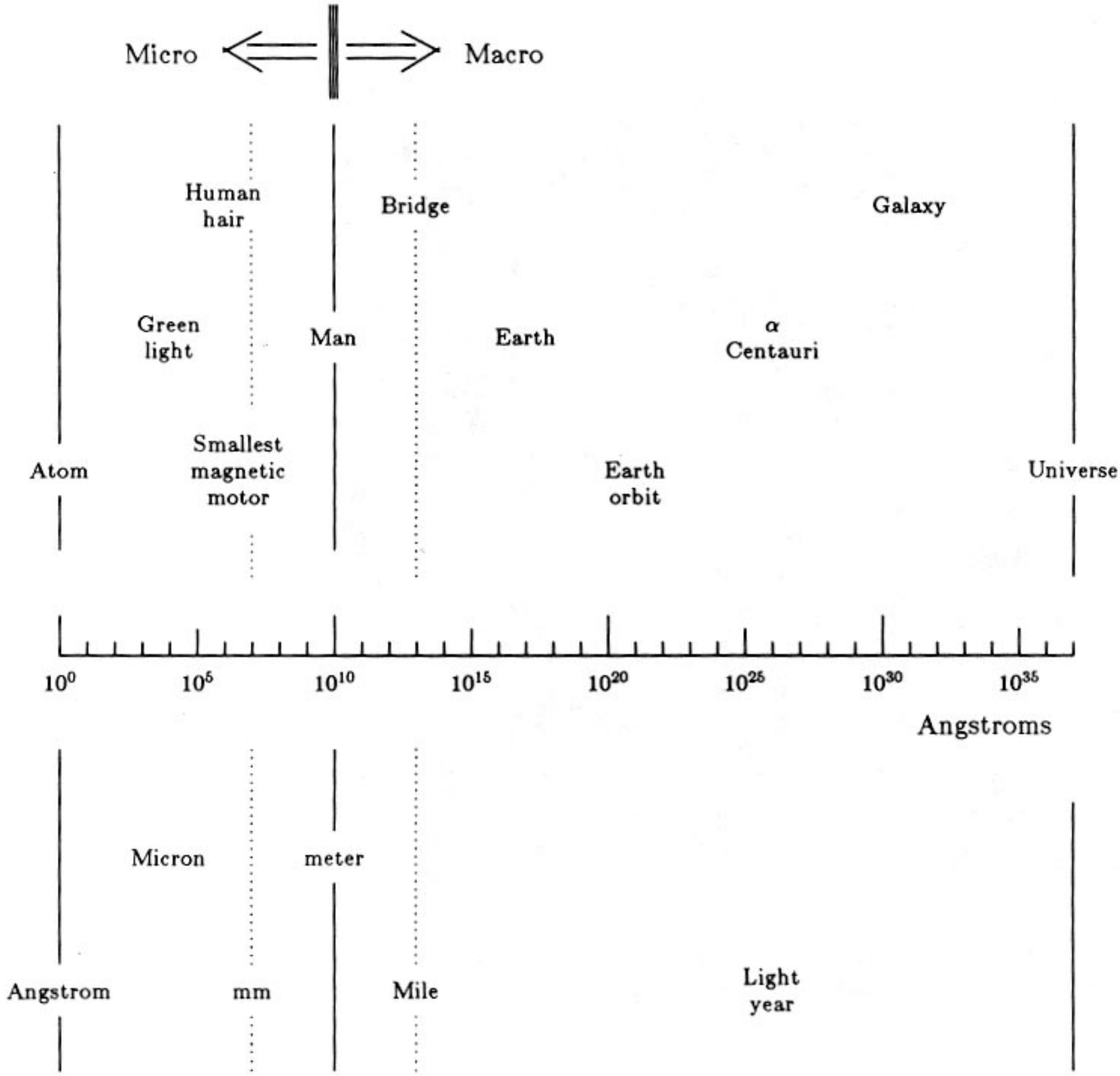


Fig. 1. The range of mechanical objects is shown plotted on a logarithmic scale. Objects range from atoms (roughly Ångstrom sized) to man (about a meter or 10^{10} Ångstroms in size) to the Universe ($\approx 10^{37}$ Ångstroms). Micromechanical systems span about a quarter of the scale. Since mechanical systems larger than a few miles will probably not be built in the near future, micro systems are the majority of the scale available for exploration.

How do things behave in micro-scale?

Ants Vs. Human

Ants	Human
Lift 10 times its own weight	Barely its own weight
Do not injure when falling	Easily injure
Some can fly	Wish to
Leg is thin comparing to body	Leg or arm are thicker

- We do not have good intuition of microscale physics. Our common sense, rules of thumb in macro world, can not apply well in micro world
- Use known physical concepts, classical physics and continuum mechanics, to dimensional analysis to reveal real behaviors in micro world.
- **Scaling analysis:**
  Most physical quantities (force, mass, volume, etc.) scale differently with dimension L.
  Assume isometric scaling (i.e., geometric similarity):

  Length ~ L
  Area ~ L^2
  volume ~ L^3

  different force scaling:

  Surface tension force ~ L ($F_\sigma = L\sigma, \sigma$: surface tension)

  Skin Friction force ~ L^2
  Heat transfer ~ L^2
  Diffusion ~ L^2
  Muscle strength ~ L^2
  Bone strength ~ L^2

  Mass ~ L^3

Isometric (by the same measure) Vs. **Allometric** (by a different measure):

Isometric: $y = bx^a$, a=1, y is proportional to x, proportional change is isometric

Allometric Formula (Georges Teissier, 1931, J. S Huxley, 1932):
It consists simply of comparing the relation of two measurements, ignoring all the complexities and detail changes in the form.

$y = bx^a$, a and b are some constant, x and y are two different dimensions

$\Rightarrow \log y = \log b + a \log x$, a$\neq$1, non-proportional change is allometric

Example 1: Isometric Case—Weight lifting

Observed: Different-sized individuals of the same species generally keep a reasonably faithful proportion—isometry

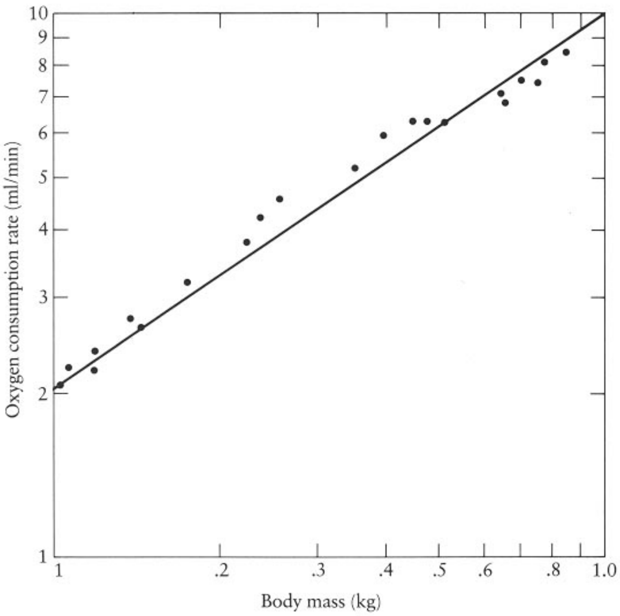
$$\mathbf{D_{arm} \sim L_{body}}$$

Assumption: Muscle stress is the same for different body size
($\sigma = F / A = constant$)

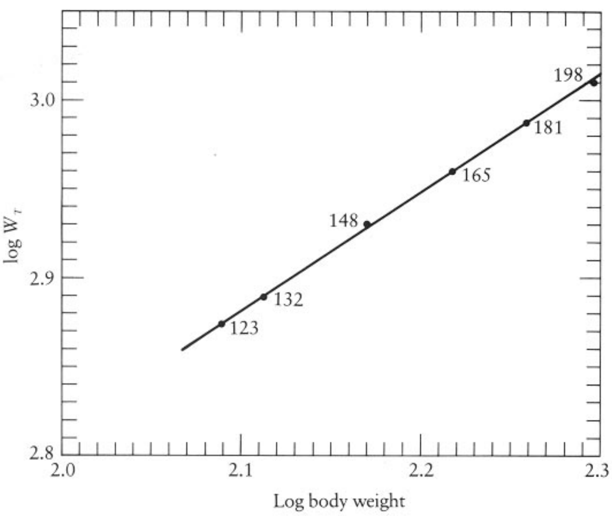
If we predict the lifting ability of a person based on weight:

$$W_{body} \sim L_{body}^3$$
$$W_{lift} \sim \sigma D_{arm}^2 \sim L_{body}^2$$
$$Thus, W_{lift} \sim (W_{body})^{2/3}$$

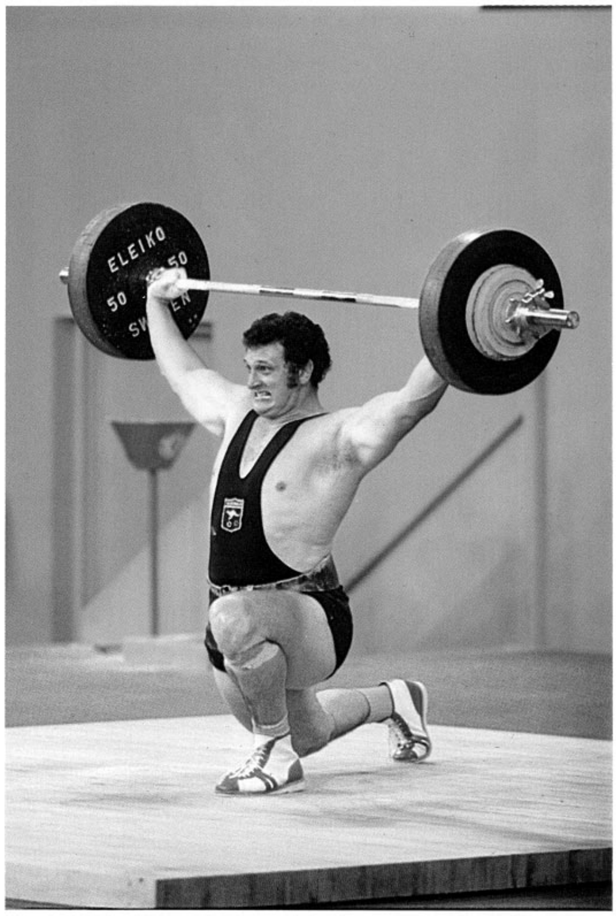
World weight-lifting records agree well!!



Rate of oxygen consumption (basal metabolic rate) for resting guinea pigs of various sizes. This rate is proportional to body mass raised to the power 0.67.



World weight-lifting records, represented by log W_T, plotted against log body weight. Here, W_T is the total weight lifted in three lifts: the press, the snatch, and the clean-and-jerk. The numbers beside each point indicate the body weight class, given in pounds.



Weight lifted in body weight classes up to 198 pounds is precisely proportional to the 0.67 power of body weight, that is, the ⅔ power of body weight in animals scaled by isometry.

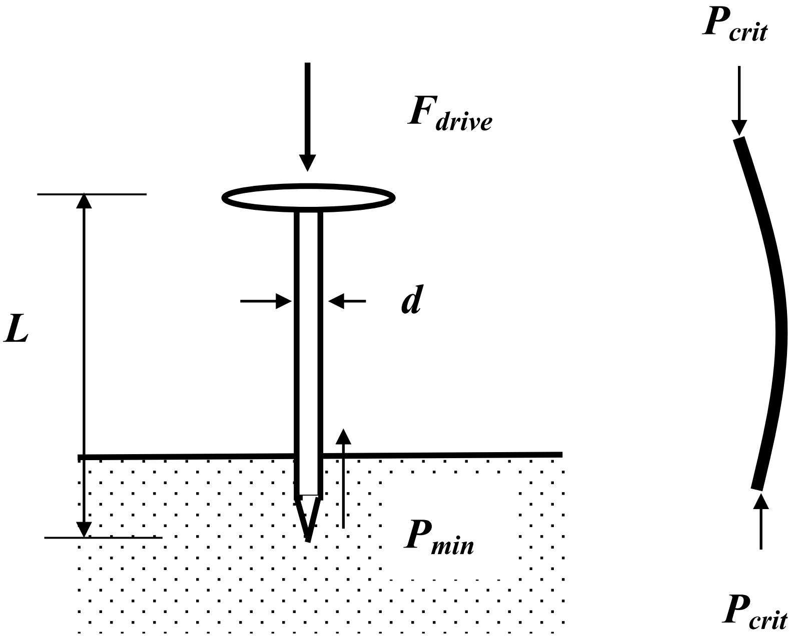
Question: Do things scale always proportionally?

No, only in limited size range or within a species, or under some special circumstances. Mostly, the size difference among different terrestrial plants and animals are allometric.

Example2: allometric case: common nails produced

Observed: for living things of greatly different sizes (difference species), proportions change with size and are allometric.

For common nails, dose **d~L**?



To drive nail in without bulking: $P_{\min} < F_{drive} < P_{crit}$

Buckling consideration:

$$P_{crit} = n\pi^2 EI / L^2$$

(n=1/4 for fix-free, 1 for pin-pin, 2 for pin-fix, 4 for fix-fix)

Critical load P_{crit}~EI/L^2, where I: moment of inertia

=> Since $\mathbf{I \sim d^4}$, => $\mathbf{P_{crit} \sim d^4/L^2}$

Friction consideration:

Minimum load (empirical+assumption): $\mathbf{P_{min} \sim \pi d}$

The case limitation: $\mathbf{P_{crit} = P_{min} \Rightarrow d^4/L^2 \sim d \Rightarrow d \sim L^{2/3}}$ **(compare to Elastic similarity** $\mathbf{d \sim L^{3/2}}$**)**

Actual data agree with the relationship!! (on size and life by McMahon and Bonner)

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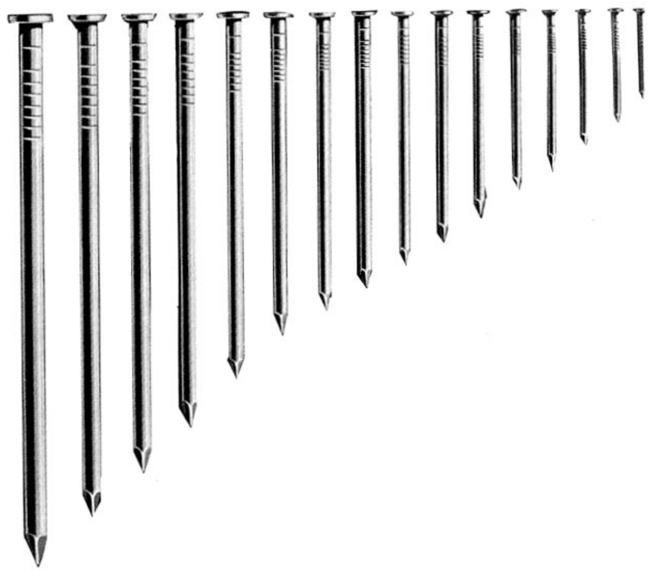
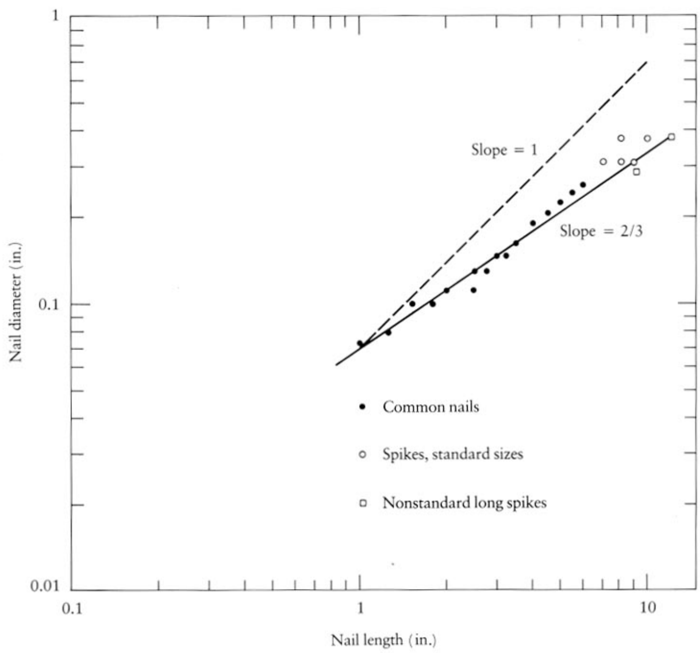


Common nails arranged by size from 60 penny (6 inches) to 2 penny (1 inch).



The ratio of nail length to nail diameter on a log–log plot, showing the allometric formula $d = 0.07l^{2/3}$. A broken line of slope 1.0, representing strict isometry, is also shown.

Why dimensional analysis is important for MEMS?

- To design or to communicate with others, we blow up the scale of the micro-devices linearly—insometry
- Intuition tends to perceive the microscale object as geometrically similar counterparts of those in our world.=>hinders design in conceptual level
- As scale reduces, certain physical quantities become significant (or negligible) compared with other quantities.

A. Inertia force:

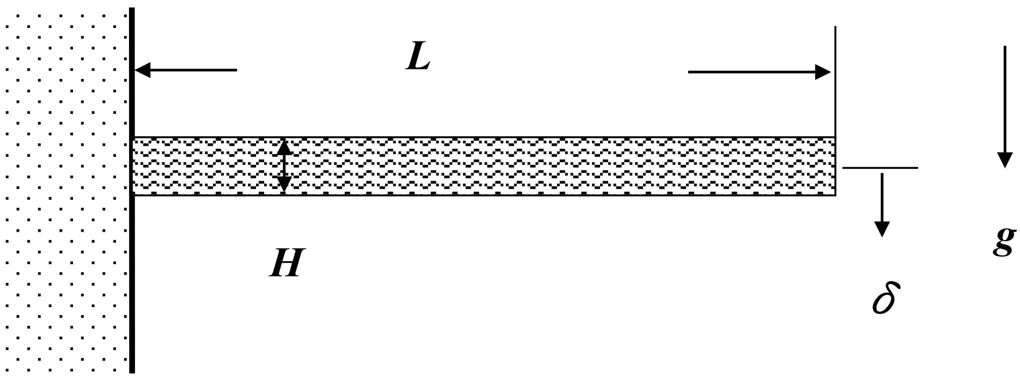
Deformation by inertia force:

Stress ~ W/A~L^3/L^2=L

Relative deformation (strain)=W/AE ~ L (size)

i.e., strain by inertia force decreases as size decreases

bending by weight:



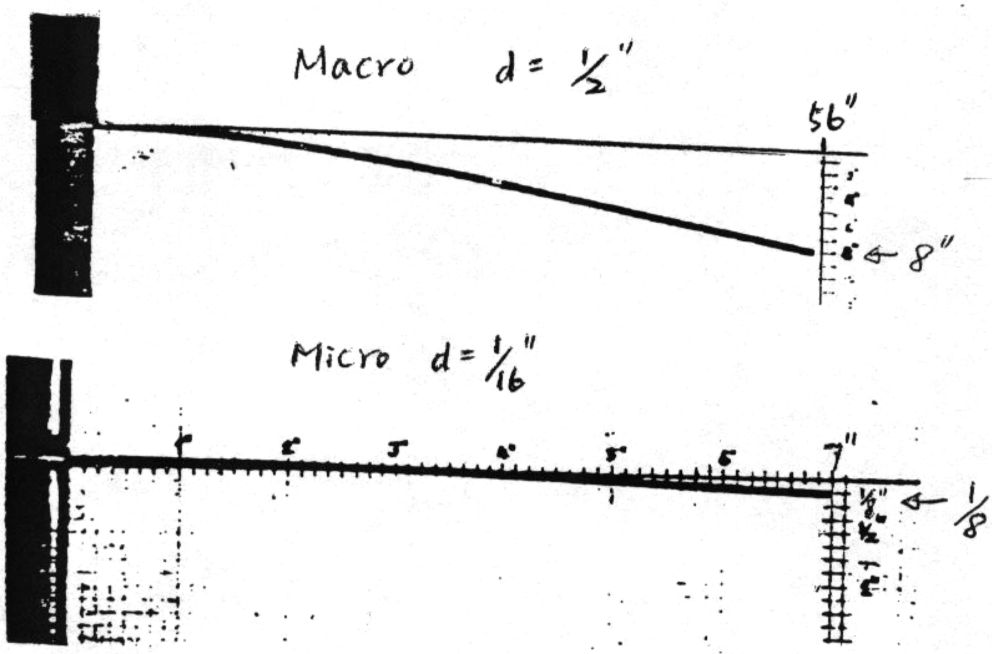
relative transverse deformation

$$\delta/L=(3/2)(\rho g/E)(L/H)^2 L \sim$$

(material properties)(aspect ratio)(size)

compare beams a and b with geometric similarity but with different absolute size:

$$\frac{\delta_a / L_a}{\delta_b / L_b} = \frac{L_a}{L_b}$$



The beam in upper picture: 56” long, 1/2” diameter
The beam in lower picture: 7” long, 1/16” diameter
(same material, same aspect ratio—isometry)

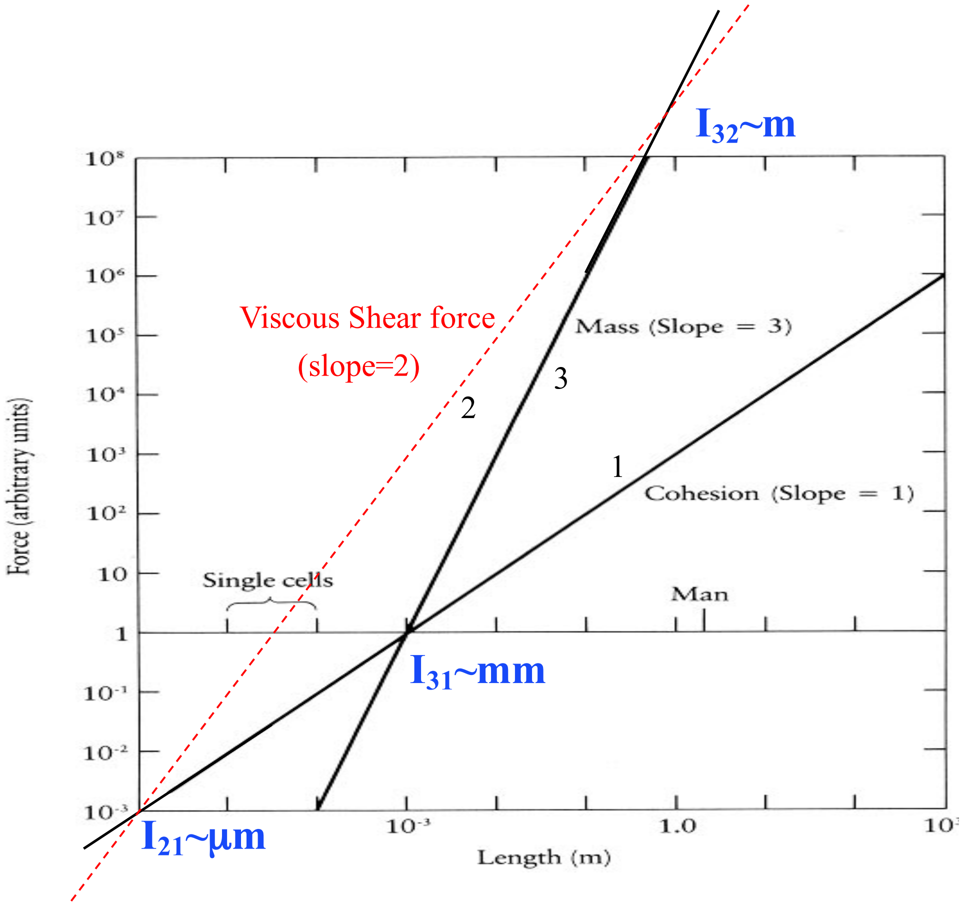
The size difference in the example is 8 times.

The relative deformation of a 300 μm-long beam (typical size for microbeams) would be 1/5000 that of the 56” long beam.

⇨ In microscale, structures appear stiffer against inertia forces.
⇨ Inertial force is generally insignificant for micro-devices. (proof mass for accelerometers; strong against shock)

B. Surface Force:

- Body force (~L^3), surface force (~L^2), and line force (~L) are significant in macro, micro, nano scale, respectively



Scaling of weight and molecular adhesion, based on empirical observation (W. went, American Scientist 1968): Froces due to mass (proportional to length cubed), shear force (roughly proportional to length squared, added by FGT), and molecular cohesion (roughly proportional to length) are shown on the same graph so they can be compared. The lines are arbitrarily allowed to intersect at a length of 1m (at I_{32}), 1 mm (at I_{31}), and 1 µm (at I_{21}), respectively.

- Stress by surface tension=surface tension force/area

$$=\gamma L/L^2 \sim L^{-1}$$

i.e., stress by surface force increases as size decreases

⇨ adhesion is a important issue in micromachining process

⇨ Taking advantage: device actuated by surface tension force

C. Locomotion:

Reynolds number characterizes the flow around an object

Re=ρVL/μ ~ (Inertial force)/(viscous force)

Where ρ: fluid density, V: velocity

L: size of object, μ: fluid viscosity

$$V \sim L^{1/2} \ (V \sim L/t, \text{ and } t \sim \sqrt{\frac{2Lm}{F}})$$

$$\Rightarrow Re \sim L^{3/2}$$

Transition from laminar to turbulent: $Re \sim 10^3\text{-}10^5$

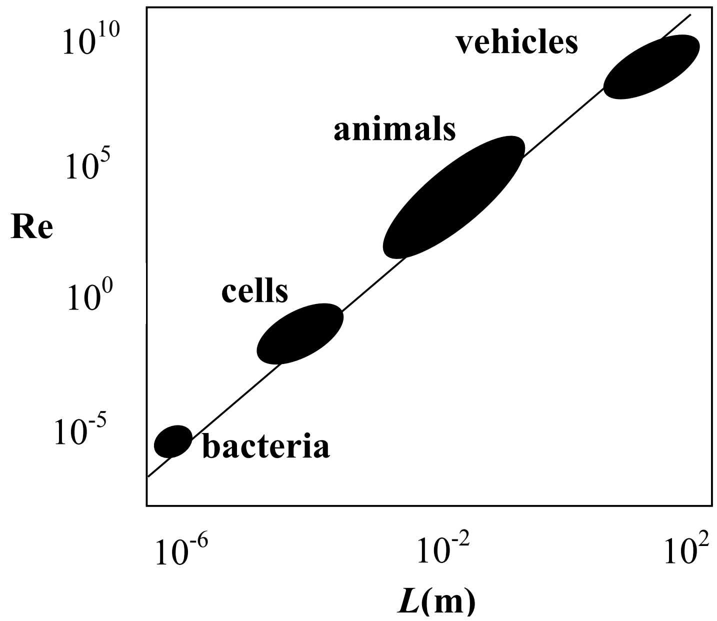
e.g. Blue whale swing at $Re \sim 10^8$

large fish swims at $Re \sim 10^5$

large birds flies at $Re \sim 10^4$

Insect flies at $Re \sim 10\text{-}10^2$

Bacteria swims at $Re \sim 10^{-6}$



Low Reynolds number locomotion:

⇨ Short stopping distance: no gliding for micro-objects

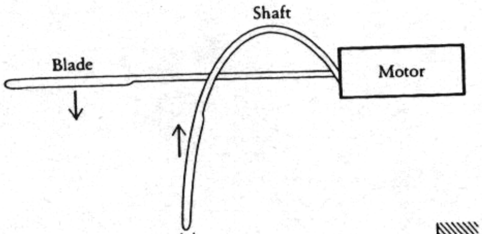
e.g. birds (e.g. pelican) diving vs. insect diving into water

⇨ Reversibility:

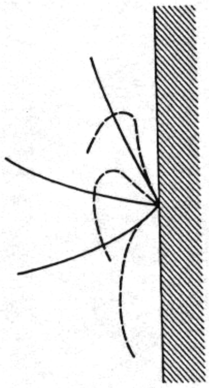
e.g. swinging rudder in mud would be ineffective

Thus use a different scheme to swim:

e.g. ciliary's propulsion, flagella propulsion

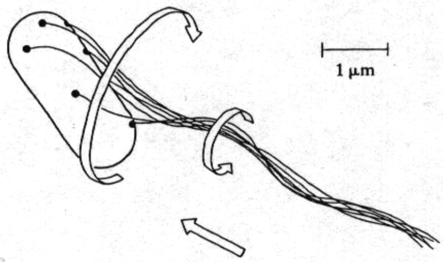
- 

(a)



(b)

A flexible oar (*a*) for rowing at low Reynolds numbers remains reasonably rigid on the thrust stroke (moving downward) but bends significantly on the return stroke. Ciliary beating (*b*) uses the flexible-oar principle. The solid curves show successive positions of the cilium beating downwards; the broken curves show the cilium on the return stroke.



a typical flagellated bacterium. One or several long, helical filaments (flagella) project from the surface of the cell. When the bacterium is swimming steadily, the separate filaments are united in a flagellar bundle, which pushes the cell along. The arrows show that the cell body and the flagellar bundle rotate in opposite directions during steady swimming. Periods of steady swimming are interrupted by periods of erratic motions in which the bundle flies apart.

Trimmer's Vertical Bracket Notation to Represent Scaling law:

For different possible forces (actually most of the forces are within those situations) he writes:

$$F = \begin{bmatrix} L^1 \\ L^2 \\ L^3 \\ L^4 \end{bmatrix}, \quad \text{L : dimension}$$

for $$a = \frac{F}{m} = \left[L^F\right]\left[L^{-3}\right] = \begin{bmatrix} L^{-2} \\ L^{-1} \\ L^0 \\ L^1 \end{bmatrix}, \text{ a: acceleration}$$

$$t = \sqrt{\frac{2xm}{F}} = \left(\left[L^1\right]\left[L^3\right]\left[L^{-F}\right]\right)^{1/2} = \begin{bmatrix} L^{1.5} \\ L^1 \\ L^{0.5} \\ L^0 \end{bmatrix}, \text{ t: time scale}$$

$$\frac{P}{V} = \frac{Fx}{tV} = \frac{\left[L^F\right]\left[L^1\right]}{\left[L^{\frac{4-F}{2}}\right]\left[L^3\right]} = \begin{bmatrix} L^{-2.5} \\ L^{-1} \\ L^{0.5} \\ L^2 \end{bmatrix}, \text{ P/V: power/volume}$$

generated/dissipated

Surface tension force=γL=$\left[L^1\right]$

Muscle force=$\left[L^2\right]$

Magnetic force: wire to wire 1. const current density =$\left[L^4\right]$

2. Constant heat flow =$\left[L^3\right]$

3. Constant temp rise =$\left[L^2\right]$

wire to magnet 1. const current density =$\left[L^3\right]$

2. Constant heat flow $=\lfloor L^{2.5} \rfloor$

3. Constant temp rise $=\lfloor L^{2} \rfloor$

Electrostatic force: $F = -\frac{1}{2}\varepsilon_0 \frac{\partial}{\partial x}[wldE^2]$ 1. constant E$=\lfloor L^{2} \rfloor$

2. E~$\lfloor L^{-0.5} \rfloor = \lfloor L^{1} \rfloor$

Reference:

1. On Size and Life, T. A. McMahon and J. T. Bonner, Scientific American Library, 1983
2. Micro Mechanics and MEMS, classic and seminar papers to 1990, William S. Trimmer, IEEE press, pp. 96-116, 1997.